

Arnold diffusion for cusp-generic nearly integrable convex systems on \mathbb{A}^3

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Abstract

Using the results of [Mar] and [GM], we prove the existence of “Arnold diffusion orbits” in cusp-generic nearly integrable *a priori* stable systems on \mathbb{A}^3 .

More precisely, we consider perturbed systems of the form $H(\theta, r) = h(r) + f(\theta, r)$, where h is a C^κ strictly convex and superlinear function on \mathbb{R}^3 and $f \in C^\kappa(\mathbb{A}^3)$, $\kappa \geq 2$. We equip $C^\kappa(\mathbb{A}^3)$ with the uniform seminorm

$$\|f\|_\kappa = \sum_{k \in \mathbb{N}^6, 0 \leq |k| \leq \kappa} \|\partial^k f\|_{C^0(\mathbb{A}^3)} \leq +\infty$$

we set $C_b^\kappa(\mathbb{A}^3) = \{f \in C^\kappa(\mathbb{A}^3) \mid \|f\|_\kappa < +\infty\}$, and we denote by \mathcal{S}^κ its unit sphere. Given a “threshold function” $\varepsilon_0 : \mathcal{S}^\kappa \rightarrow [0, +\infty[$, we define the associated ε_0 -ball as

$$\mathcal{B}^\kappa(\varepsilon_0) := \{\varepsilon \mathbf{f} \mid \mathbf{f} \in \mathcal{S}^\kappa, \varepsilon \in]0, \varepsilon_0(\mathbf{f})[\},$$

so that $\mathcal{B}^\kappa(\varepsilon_0)$ is open in $C_b^\kappa(\mathbb{A}^3)$ when ε_0 is lower semicontinuous.

Given h as above, an energy $\mathbf{e} > \text{Min } h$ and a finite family of arbitrary open sets O_i in \mathbb{R}^3 intersecting $h^{-1}(\mathbf{e})$, we prove the existence of a lower semicontinuous threshold function ε_0 , positive on an open dense subset of \mathcal{S}^κ , such that for f in an open dense subset of the associated ball $\mathcal{B}^\kappa(\varepsilon_0)$, the system $H = h + f$ admits orbits intersecting each open set $\mathbb{T}^3 \times O_i \subset \mathbb{A}^3$.

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1 The main results

We denote by $\mathbb{A}^n = \mathbb{T}^n \times \mathbb{R}^n$ the cotangent bundle of the torus \mathbb{T}^n . This paper is the last step of a geometrical proof of the existence of Arnold diffusion for a “large subset” of perturbations of strictly convex integrable systems on \mathbb{A}^3 . It relies on the results of [Mar] and [GM]. Two different approaches are developed in [KZ] and [C].

1. Before giving a precise statement, let us quote the initial formulation of the diffusion conjecture by V.I. Arnold, from [Arn94]. “Consider a generic analytic Hamiltonian system close to an integrable one:

$$H = H_0(p) + \varepsilon H_1(p, q, \varepsilon)$$

where the perturbation H_1 is 2π -periodic in the angle variables (q_1, \dots, q_n) and where the nonperturbed Hamiltonian function H_0 depends on the action variables (p_1, \dots, p_n) generically. Let n be greater than 2.

Conjecture. For any two points p', p'' on the connected level hypersurface of H_0 in the action space, there exist orbits connecting an arbitrary small neighborhood of the torus $p = p'$ with an arbitrary small neighborhood of the torus $p = p''$, provided that $\varepsilon > 0$ is sufficiently small and that H_1 is generic.”

2. The formulation of the conjecture is rather imprecise and we first have to clarify our framework. We are concerned with the case $n = 3$ only and we restrict ourselves to finitely differentiable systems. Moreover, we adopt a setting close to that introduced by Mather in [Mat04], which we now describe with our usual notation.

Let (θ, r) be the angle-action coordinates on \mathbb{A}^3 , and let $\lambda = \sum_{i=1}^3 r_i d\theta_i$ be the Liouville form of \mathbb{A}^3 . The Hamiltonian systems we consider have the form

$$H(\theta, r) = h(r) + f(\theta, r), \quad (\theta, r) \in \mathbb{A}^3, \quad (1)$$

where the unperturbed part $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a C^κ ($\kappa \geq 2$) Tonelli Hamiltonian, that is, h is strictly convex with superlinear growth at infinity. Here we therefore relax the genericity condition initially imposed by Arnold.

We want to find C^κ Hamiltonians H which admit *diffusion orbits* intersecting prescribed open sets in their energy level. More precisely, we start with a Tonelli Hamiltonian h and fix an energy $\mathbf{e} > \text{Min } h$, which is therefore a regular value of h whose level set $h^{-1}(\mathbf{e})$ is diffeomorphic to S^2 . We then fix a finite collection of arbitrary open sets $(O_i)_{1 \leq i \leq i_*}$ in \mathbb{R}^3 , which intersect $h^{-1}(\mathbf{e})$. Given $H \in C^\kappa(\mathbb{A}^3, \mathbb{R})$, a diffusion orbit associated with these data is an orbit of the system generated by H which intersects each open set $\mathbb{T}^3 \times O_i \subset \mathbb{A}^3$.

Our problem is to prove the existence of a “large” set of perturbations f for which (1) possesses such diffusion orbits. We equip $C^\kappa(\mathbb{A}^3, \mathbb{R})$ with the uniform seminorm

$$\|f\|_\kappa = \text{Max}_{0 \leq |k| \leq \kappa} \text{Sup}_{x \in \mathbb{A}^3} \partial^k f(x) \leq +\infty, \quad (2)$$

and we set

$$C_b^\kappa(\mathbb{A}^3, \mathbb{R}) = \{f \in C^\kappa(\mathbb{A}^3, \mathbb{R}) \mid \|f\|_\kappa < +\infty\}, \quad (3)$$

so that $(C_b^\kappa(\mathbb{A}^3, \mathbb{R}), \|\cdot\|_\kappa)$ is a Banach algebra. Let \mathcal{S}^κ and $B^\kappa(\rho)$ stand for the unit sphere and the ball with radius ρ centered at 0 in $C_b^\kappa(\mathbb{A}^3, \mathbb{R})$. Given a “threshold function” $\varepsilon_0 : \mathcal{S}^\kappa \rightarrow \mathbb{R}$, one introduces the associated generalized ball:

$$\mathcal{B}^\kappa(\varepsilon_0) := \{\varepsilon u \mid u \in \mathcal{S}^\kappa, \varepsilon \in]0, \varepsilon_0(u)[\}. \quad (4)$$

Observe that $\mathcal{B}^\kappa(\varepsilon_0)$ is open in $C_b^\kappa(\mathbb{A}^3, \mathbb{R})$ when ε_0 is lower-semicontinuous.

3. The main result of this paper is the following.

Theorem 1. *Consider a C^κ integrable Tonelli Hamiltonian h on \mathbb{A}^3 . Fix $\mathbf{e} > \text{Min } h$ together with a finite family of arbitrary open sets O_1, \dots, O_m which intersect $h^{-1}(\mathbf{e})$. Then for $\kappa \geq \kappa_0$ large enough, there exists a lower-semicontinuous function*

$$\varepsilon_0 : \mathcal{S}^\kappa \rightarrow \mathbb{R}^+$$

with positive values on an open dense subset of \mathcal{S}^κ such that the subset of all $f \in \mathcal{B}^\kappa(\varepsilon_0)$ for which the system

$$H(\theta, r) = h(r) + f(\theta, r) \tag{5}$$

admits an orbit which intersects each $\mathbb{T}^3 \times O_k$ is open and dense in $\mathcal{B}^\kappa(\varepsilon_0)$.

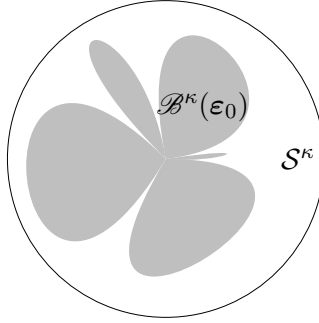


Figure 1: A generalized ball

4. Our approach consists in proving first the existence of a “geometric skeleton” for diffusion and then to use this skeleton to produce the diffusion orbits. Let us informally describe our method (the precise definitions were introduced in [Mar] and [GM], they are recalled in Section 2).

The main objects constituting the skeleton are 3-dimensional invariant cylinders with boundary, diffeomorphic to $\mathbb{T}^2 \times [0, 1]$, which are moreover normally hyperbolic and satisfy several additional properties, and which we call *admissible cylinders*. We also have to consider admissible *singular* cylinders. These cylinders and singular cylinders contain particular 2-dimensional invariant tori, which we call *essential tori*, and admit *homoclinic correspondences* coming from the homoclinic intersections of the essential tori.

Finally, we define an *admissible chain*, as a finite family $(\mathcal{C}_k)_{1 \leq k \leq k_*}$ of admissible cylinders or singular cylinders, with heteroclinic connections from \mathcal{C}_k to \mathcal{C}_{k+1} , $1 \leq k \leq k_* - 1$, which satisfy additional dynamical conditions.

The main result of [Mar] is the following.

Theorem [Mar]. *Consider a C^κ integrable Tonelli Hamiltonian h on \mathbb{A}^3 . Fix $\mathbf{e} > \text{Min } h$ and a finite family of open sets O_1, \dots, O_m which intersect $h^{-1}(\mathbf{e})$. Fix $\delta > 0$. Then for $\kappa \geq \kappa_0$ large enough, there exists a lower-semicontinuous function*

$$\varepsilon_0 : \mathcal{S}^\kappa \rightarrow \mathbb{R}^+$$

with positive values on an open dense subset of \mathcal{S}^κ such that for $f \in \mathcal{B}^\kappa(\varepsilon_0)$ the system

$$H(\theta, r) = h(r) + f(\theta, r) \tag{6}$$

admits an admissible chain of cylinders and singular cylinders such that each open set $\mathbb{T}^3 \times O_k$ contains the δ -neighborhood in \mathbb{A}^3 of some essential torus of the chain.

Still, admissible chains need not admit diffusion orbits drifting along them. In [GM] is introduced the more refined notion of *good chains of cylinders*. A δ -admissible orbit for a good chain is an orbit which intersects the δ -neighborhood (in \mathbb{A}^3) of any essential torus of the chain. The main result of [GM] is the following.

Theorem [GM]. *Let H be a C^2 proper Hamiltonian on \mathbb{A}^3 and let \mathbf{e} be a regular value of H . Then, for any good chain of cylinders contained in $H^{-1}(\mathbf{e})$ and for any $\delta > 0$, there exists a δ -admissible orbit for the chain.*

5. Taking the previous two results for granted, Theorem 1 is an easy consequence of the following perturbative result, whose proof constitutes the main part of the paper.

Theorem 2. *Let H be a C^κ proper Hamiltonian on \mathbb{A}^3 and let \mathbf{e} be a regular value of H . Fix $\delta > 0$ and assume that H admits an admissible chain $(\mathcal{C}_k)_{1 \leq k \leq k_*}$. Then for any $\alpha > 0$ there exists a Hamiltonian $\mathcal{H} \in C^\kappa(\mathbb{A}^3)$ with*

$$\|H - \mathcal{H}\|_\kappa < \alpha \quad (7)$$

such that $(\mathcal{C}_k)_{1 \leq k \leq k_}$ is a good chain at energy \mathbf{e} for \mathcal{H} , such that each open set $\mathbb{T}^3 \times O_k$ contains the δ -neighborhood in \mathbb{A}^3 of some essential torus.*

6. We recall the necessary definitions from [Mar] and [GM] in Section 2. We state in Section 3 a perturbative result for the characteristic foliations of the stable and unstable manifolds of a normally hyperbolic manifold, which is the main ingredient for the proof of Theorem 2. In Section 4 we prove Theorem 2, from which we deduce Theorem 1 thanks to the previous two results of [Mar, GM]. We recall some necessary results on normally hyperbolic manifolds in Appendix A.

2 The setting

The Hamiltonian vector field associated with a C^2 function H will be denoted by X_H and its Hamiltonian flow, when defined, by Φ_H .

2.1 Normally hyperbolic annuli and cylinders

We introduce the main objects of our construction, that is, normally hyperbolic 3-dimensional cylinders and singular cylinders with boundary. We refer to [Cha04, Ber10a] for direct presentations of the normal hyperbolicity of manifolds with boundary. Here we will recall the definitions of [Mar] (to which we refer for more details), which take advantage of the existence of invariant 4-dimensional symplectic annuli containing the cylinders in their relative interior. These annuli will moreover play an essential role in the definition of the intersection conditions in the next section.

1. A 4-annulus will be a C^p manifold C^p diffeomorphic to \mathbb{A}^2 , with $p \geq 2$. A *singular annulus* will be a (4-dimensional) C^1 manifold C^1 -diffeomorphic to $\mathbb{T} \times]0, 1[\times \mathcal{Y}$, where \mathcal{Y} is (any realization of) the sphere S^2 minus three points.

2. A C^p cylinder is a C^p manifold C^p -diffeomorphic to $\mathbb{T}^2 \times [0, 1]$, so that a cylinder is compact and its boundary has two components diffeomorphic to \mathbb{T}^2 . A *singular cylinder*

is a C^1 manifold C^1 -diffeomorphic to $\mathbb{T} \times \mathbf{Y}$, where \mathbf{Y} is (any realization of) the sphere S^2 minus three open discs with nonintersecting closures. Our singular cylinders will moreover be of class C^p , $p \geq 2$, in large neighborhoods of their boundary (which admits three components, diffeomorphic to \mathbb{T}^2).

3. We endow now \mathbb{A}^3 with its standard symplectic form Ω , and we assume that $X = X_H$ is the vector field generated by $H \in C^\kappa(\mathbb{A}^3)$, $\kappa \geq 2$. Following [Mar], we say that an invariant 4-annulus $\mathcal{A} \subset \mathbb{A}^3$ for X is *normally hyperbolic* when there exist

- an open subset O of \mathbb{A}^3 containing \mathcal{A} ,
- an embedding $\Psi : O \rightarrow \mathbb{A}^2 \times \mathbb{R}^2$ whose image has compact closure, such that $\Psi_*\Omega$ continues to a symplectic form $\overline{\Omega}$ on $\mathbb{A}^2 \times \mathbb{R}^2$ which satisfies (Appendix A (85)),
- a vector field \mathcal{V} on $\mathbb{A}^2 \times \mathbb{R}^2$ satisfying the assumptions of the normally hyperbolic persistence theorem, in particular (83), together with those of the symplectic normally hyperbolic theorem (Appendix A) for the form $\overline{\Omega}$, such that, with the notation of this theorem:

$$\Psi(\mathcal{A}) \subset \text{Ann}(\mathcal{V}) \quad \text{and} \quad \Psi_*X(x) = \mathcal{V}(x), \quad \forall x \in O. \quad (8)$$

Such an annulus \mathcal{A} is therefore of class C^p and symplectic. We define normally hyperbolic singular 4-annuli, with in this case $p = 1$. One easily checks that annuli and singular annuli are uniformly normally hyperbolic in the usual sense. In particular, they admit well-defined stable, unstable, center-stable and center-unstable manifolds. The stable and unstable manifolds are coisotropic and their characteristic foliations coincide with their center-stable and center-unstable foliations

4. With the same assumptions, let \mathbf{e} be a regular value of H . Here we say for short that an invariant cylinder (with boundary) $\mathcal{C} \subset H^{-1}(\mathbf{e})$ for X_H is *normally hyperbolic in $H^{-1}(\mathbf{e})$* when there exists an invariant normally hyperbolic *symplectic* 4-annulus \mathcal{A} for X_H , such that $\mathcal{C} \subset \mathcal{A} \cap H^{-1}(\mathbf{e})$. Any such \mathcal{A} is said to be *associated with \mathcal{C}* .

We say for short that a singular cylinder $\mathcal{C}_\bullet \subset H^{-1}(\mathbf{e})$ is invariant for X_H when it is invariant together with its critical circles. We say that \mathcal{C}_\bullet is *normally hyperbolic in $H^{-1}(\mathbf{e})$* when there is an invariant normally hyperbolic *symplectic* singular annulus \mathcal{A}_\bullet for X_H such that $\mathcal{C}_\bullet \subset \mathcal{A}_\bullet \cap H^{-1}(\mathbf{e})$. Any such \mathcal{A}_\bullet is said to be *associated with \mathcal{C}_\bullet* .

One immediately sees that normally hyperbolic invariant cylinders or singular cylinders, contained in $H^{-1}(\mathbf{e})$, admit well-defined 4-dimensional stable and unstable manifolds with boundary, also contained in $H^{-1}(\mathbf{e})$, together with their center-stable and center-unstable foliations. The stable and unstable manifolds of the complement in a singular cylinder of its critical circles are C^p .

5. A normally hyperbolic cylinder admits C^1 characteristic projections $\Pi^\pm : W^\pm(\mathcal{C}) \rightarrow \mathcal{C}$ (since the invariant manifolds $W^\pm(\mathcal{A})$ of are C^p with $p \geq 2$), it satisfies the λ -lemma (as stated in [GM]) and one easily proves that it admits a Φ_H -invariant Radon measure $\mu_{\mathcal{C}}$, positive on its open sets. It is therefore *tame* in the sense of [GM]. A normally hyperbolic singular cylinder admits C^0 characteristic projections, which are C^1 in a large neighborhood of its boundary, it also satisfies the λ -lemma and admits a Φ_H -invariant Radon measure $\mu_{\mathcal{C}}$, positive on its open sets.

6. Let $a < b$ be fixed. Let us introduce the notation:

$$\mathbf{A} := \mathbf{A}(a, b) = \mathbb{T} \times [a, b], \quad \partial_\bullet \mathbf{A} = \mathbb{T} \times \{a\}, \quad \partial^\bullet \mathbf{A} = \mathbb{T} \times \{b\}.$$

A *twist section* for a cylinder $\mathcal{C} \subset H^{-1}(\mathbf{e})$ is a global 2-dimensional transverse section $\Sigma \subset \mathcal{C}_\varepsilon$, image of an exact-symplectic embedding $j_\Sigma : \mathbf{A}$, such that the associated Poincaré return map is a twist map in the j_Σ -induced coordinates on \mathbf{A} . Denote by $\mathbf{Ess}(\varphi)$ the set of essential invariant circles of φ (that is, whose inverse image by j_Σ is homotopic to the base). By the Birkhoff theorem, these circles are Lipschitzian graphs over the base. One requires moreover that the boundaries of Σ are accumulation points of $\mathbf{Ess}(\varphi)$.

Note that $\partial\mathcal{C} \cap \Sigma = \partial_\bullet\mathbf{A} \cup \partial^\bullet\mathbf{A}$. We define $\partial_\bullet\mathcal{C}$ and $\partial^\bullet\mathcal{C}$ as the components of $\partial\mathcal{C}$ which contain $j_\Sigma(\partial_\bullet\mathbf{A})$ and $j_\Sigma(\partial^\bullet\mathbf{A})$ respectively.

7. A generalized twist section for a singular cylinder is a singular 2 annulus which admits a continuation to a 2-annulus, on which the Poincaré return map continues to a twist map (see [Mar, GM]).

2.2 Intersection conditions, gluing condition, and admissible chains

Let H be a proper C^2 Hamiltonian function on \mathbb{A}^3 and fix a regular value \mathbf{e} .

1. Oriented cylinders. We say that a cylinder \mathcal{C} is *oriented* when an order is prescribed on the two components of its boundary. We denote the first one by $\partial_\bullet\mathcal{C}$ and the second one by $\partial^\bullet\mathcal{C}$.

2. The homoclinic condition (FS1). A compact invariant cylinder $\mathcal{C} \subset H^{-1}(\mathbf{e})$ with twist section Σ and associated invariant symplectic 4-annulus \mathcal{A} satisfies condition (FS1) when there exists a 5-dimensional submanifold $\Delta \subset \mathbb{A}^3$, transverse to X_H such that:

- there exist 4-dimensional submanifolds $\mathcal{A}^\pm \subset W^\pm(\mathcal{A}) \cap \Delta$ such that the restrictions to \mathcal{A}^\pm of the characteristic projections $\Pi^\pm : W^\pm(\mathcal{A}) \rightarrow \mathcal{A}$ are diffeomorphisms, whose inverses we denote by $j^\pm : \mathcal{A} \rightarrow \mathcal{A}^\pm$;
- there exists a continuation \mathcal{C}_* of \mathcal{C} such that $\mathcal{C}_*^\pm = j^\pm(\mathcal{C}_*)$ have a nonempty intersection, transverse in the 4-dimensional manifold $\Delta_\mathbf{e} := \Delta \cap H^{-1}(\mathbf{e})$; so that $\mathcal{I}_* := \mathcal{C}_*^+ \cap \mathcal{C}_*^-$ is a 2-dimensional submanifold of $\Delta_\mathbf{e}$;
- the projections $\Pi^\pm(\mathcal{I}_*) \subset \mathcal{C}_*$ are 2-dimensional transverse sections of the vector field X_H restricted to \mathcal{C}_* , and the associated Poincaré maps $P^\pm : \Pi^\pm(\mathcal{I}_*) \rightarrow \Sigma_k$ are diffeomorphisms.

We then say that

$$\psi = P^+ \circ \Pi^+ \circ j^- \circ (P^-)^{-1} : \Sigma \rightarrow \Sigma_* \quad (9)$$

(where Σ_* is a continuation of Σ), is the *homoclinic map* attached to \mathcal{C} . Note that ψ is a Hamiltonian diffeomorphism on its image.

3. The heteroclinic condition (FS2). A pair $(\mathcal{C}_0, \mathcal{C}_1)$ of compact invariant oriented cylinders with twist sections Σ_0, Σ_1 and associated invariant symplectic 4-annuli $(\mathcal{A}_0, \mathcal{A}_1)$ satisfies condition (FS2) when there exists a 5-dimensional submanifold $\Delta \subset \mathbb{A}^3$, transverse to X_H such that:

- there exist 4-dimensional submanifolds $\widetilde{\mathcal{A}}_0^- \subset W^-(\mathcal{A}_0) \cap \Delta$ and $\widetilde{\mathcal{A}}_1^+ \subset W^+(\mathcal{A}_1) \cap \Delta$ such that $\Pi_0^-|_{\widetilde{\mathcal{A}}_0^-}$ and $\Pi_1^+|_{\widetilde{\mathcal{A}}_1^+}$ are diffeomorphisms on their images $\widetilde{\mathcal{A}}_0^-, \widetilde{\mathcal{A}}_1^+$, which we require to be neighborhoods of the boundaries $\partial^\bullet\mathcal{C}_0$ and $\partial_\bullet\mathcal{C}_1$ in \mathcal{A}_0 and \mathcal{A}_1 respectively, we denote their inverses by j_0^- and j_1^+ ;

- there exist neighborhoods $\tilde{\mathcal{C}}_0$ and $\tilde{\mathcal{C}}_1$ of $\partial^\bullet \mathcal{C}^0$ and $\partial_\bullet \mathcal{C}^1$ in continuations of the initial cylinders, such that $\tilde{\mathcal{C}}_0^- = j_0^-(\tilde{\mathcal{C}}_0)$ and $\tilde{\mathcal{C}}_1^+ = j_1^+(\tilde{\mathcal{C}}_1)$ intersect transversely in the 4-dimensional manifold $\Delta_{\mathbf{e}} := \Delta \cap H^{-1}(\mathbf{e})$, let \mathcal{I}_* be this intersection;
- the projections $\Pi_0^-(\mathcal{I}_*) \subset \mathcal{C}$ and $\Pi_1^+(\mathcal{I}_*) \subset \mathcal{C}$ are 2-dimensional transverse sections of the vector field X_H restricted to $\tilde{\mathcal{C}}_0$ and $\tilde{\mathcal{C}}_1$, and the Poincaré maps $P_0 : \Pi_0^-(\mathcal{I}_*) \rightarrow \Sigma_0$ and $P_1 : \Pi_1^+(\mathcal{I}_*) \rightarrow \Sigma_1$ are diffeomorphisms (where Σ_I stands for Poincaré sections in the neighborhoods $\tilde{\mathcal{C}}_i$).

We then say that

$$\psi = P_1 \circ \Pi^+ \circ j^- \circ (P_0)^{-1} : \Sigma_0 \rightarrow \Sigma_1 \quad (10)$$

is the *heteroclinic map* attached to \mathcal{C} (which is not uniquely defined).

4. The homoclinic condition (PS1). Consider an invariant cylinder $\mathcal{C} \subset H^{-1}(\mathbf{e})$ with twist section Σ and attached Poincaré return map φ , so that $\Sigma = j_\Sigma(\mathbb{T} \times [a, b])$, where j_Σ is exact-symplectic. Define $\mathbf{Tess}(\mathcal{C})$ as the set of all invariant tori generated by the previous circles under the action on the Hamiltonian flow (so each element of $\mathbf{Tess}(\mathcal{C})$ is a Lipschitzian Lagrangian torus contained in \mathcal{C}). The elements of $\mathbf{Tess}(\mathcal{C})$ are said to be *essential tori*.

We say that an invariant cylinder \mathcal{C} with associated invariant symplectic 4-annulus \mathcal{A} satisfies the *partial section property* (PS) when there exists a 5-dimensional submanifold $\Delta \subset \mathbb{A}^3$, transverse to X_H such that:

- there exist 4-dimensional submanifolds $\mathcal{A}^\pm \subset W^\pm(\mathcal{A}) \cap \Delta$ such that the restrictions to \mathcal{A}^\pm of the characteristic projections $\Pi^\pm : W^\pm(\mathcal{A}) \rightarrow \mathcal{A}$ are diffeomorphisms, whose inverses we denote by $j^\pm : \mathcal{A} \rightarrow \mathcal{A}^\pm$;
- there exist conformal exact-symplectic diffeomorphisms

$$\Psi^{\text{ann}} : \mathcal{O}^{\text{ann}} \rightarrow \mathcal{A}, \quad \Psi^{\text{sec}} : \mathcal{O}^{\text{sec}} \rightarrow \Delta_{\mathbf{e}} := \Delta \cap H^{-1}(\mathbf{e}) \quad (11)$$

where \mathcal{O}^{ann} and \mathcal{O}^{sec} are neighborhoods of the zero section in $T^*\mathbb{T}^2$ endowed with the conformal Liouville form $a\lambda$ for a suitable $a > 0$;

- each torus $\mathcal{T} \in \mathbf{Tess}(\mathcal{C})$ is contained in some \mathcal{C} and the image $\Psi^{\text{ann}}(\mathcal{T})$ is a Lipschitz graph over the base \mathbb{T}^2 ;
- for each such torus \mathcal{T} , setting $\mathcal{T}^\pm := j^\pm(\mathcal{T}) \subset \Delta_{\mathbf{e}}$, the images $\Psi^{\text{sec}}(\mathcal{T}^\pm)$ are Lipschitz graphs over the base \mathbb{T}^2 .

5. Bifurcation condition. See [Mar] for the definitions and assumptions. The condition we state involves the s -averaged system along a simple resonance circle, it will be translated in the following as an intrinsic condition. With the notation of [Mar]: for any $r^0 \in B$, the derivative $\frac{d}{dr}(m^*(r) - m^{**}(r))$ does not vanish. This immediately yields transverse heteroclinic intersection properties for the intersections of the corresponding cylinders.

6. The gluing condition (G). A pair $(\mathcal{C}_0, \mathcal{C}_1)$ of compact invariant oriented cylinders satisfies condition (G) when they are contained in a invariant cylinder and satisfy

- $\partial^\bullet \mathcal{C}_0 = \partial_\bullet \mathcal{C}_1$ is a dynamically minimal invariant torus that we denote by \mathcal{T} ,
- $W^-(\mathcal{T})$ and $W^+(\mathcal{T})$ intersect transversely in $H^{-1}(\mathbf{e})$.

7. Admissible chains. A finite family of compact invariant oriented cylinders $(\mathcal{C}_k)_{1 \leq k \leq k_*}$ is an *admissible chain* when each cylinder satisfies either (FS1) or (PS1) and, for k in $\{1, \dots, k_* - 1\}$, the pair $(\mathcal{C}_k, \mathcal{C}_{k+1})$ satisfies either (FS2) or (G), or corresponds to a bifurcation point.

2.3 Good cylinders and good chains

This section is dedicated to the additional conditions introduced in [GM] which produce orbits drifting along a chain.

1. Special twist maps. We say that a twist map φ of \mathbf{A} is *special* when it does not admit any essential invariant circle with rational rotation number and when moreover every element of $\mathbf{Ess}(\varphi) \setminus \partial^\bullet \mathbf{A}$ is either the upper boundary of a Birkhoff zone, or accumulated from below (in the Hausdorff topology) by a sequence of elements of $\mathbf{Ess}(\varphi)$.

2. The homoclinic correspondence. We first define the *transverse homoclinic intersection* of a normally hyperbolic cylinder $\mathcal{C} \subset H^{-1}(\mathbf{e})$ as the set

$$\text{Homt}(\mathcal{C}) \subset W^+(\mathcal{C}) \cap W^-(\mathcal{C}) \quad (12)$$

formed by the points ξ such that

$$W^-(\Pi^-(\xi)) \cap_\xi W^+(\mathcal{C}) \quad \text{and} \quad W^+(\Pi^+(\xi)) \cap_\xi W^-(\mathcal{C}), \quad (13)$$

where \cap_ξ stands for “intersects transversely at ξ relatively to $H^{-1}(\mathbf{e})$.”

• Assume that \mathcal{C} admits a twist section $\Sigma = j_\Sigma(\mathbf{A})$ and identify \mathbf{A} with Σ . A *homoclinic correspondence* associated with these data is a family of C^1 local diffeomorphisms of Σ :

$$\psi = (\psi_i)_{i \in I}, \quad \psi_i : \text{Dom } \psi_i \subset \Sigma \rightarrow \text{Im } \psi_i \subset \Sigma, \quad (14)$$

where $\text{Dom } \psi_i$ and $\text{Im } \psi_i$ are open subsets of Σ , for which there exist is a family of C^1 local diffeomorphisms of \mathcal{C} :

$$S = (S_i)_{i \in I}, \quad S_i : \text{Dom } S_i \subset \mathcal{C} \rightarrow \text{Im } S_i \subset \mathcal{C} \quad (15)$$

where $\text{Dom } S_i$ and $\text{Im } S_i$ are open subsets of \mathcal{C} , such that for all $i \in I$:

- there exists a C^1 function $\tau_i : \text{Dom } \psi_i \rightarrow \mathbb{R}$ such that

$$\forall x \in \text{Dom } \psi_i, \quad \Phi_H^{\tau_i(x)}(x) \in \text{Dom } S_i \quad \text{and} \quad \psi_i(x) = S_i\left(\Phi_H^{\tau_i(x)}(x)\right); \quad (16)$$

- there is an open subset $S_i \subset \text{Dom } S_i$, with full measure in $\text{Dom } S_i$, such that

$$\forall y \in S_i, \quad W^-(y) \cap W^+(S_i(y)) \cap \text{Homt}(\mathcal{C}) \neq \emptyset. \quad (17)$$

We say that a family S satisfying the previous properties is *associated* with ψ .

Homoclinic correspondences are not uniquely defined and the domains $\text{Dom } \psi_i$ (resp. $\text{Dom } S_i$) are not necessarily pairwise disjoint. The index set I is non countable in general. In the following we indifferently consider our homoclinic correspondences as defined on Σ or on \mathbf{A} . The following additional definition is necessary to produce δ -admissible orbits.

- Let \mathcal{C} be a good cylinder with twist section $\Sigma = j_\Sigma(\mathbf{A})$ and a homoclinic correspondence $\psi : \mathbf{A} \hookrightarrow$. Fix $\delta > 0$. We say that ψ is δ -bounded when for each essential circle $\Gamma \in \mathbf{Ess}(\varphi)$

$$\text{dist} \left(\text{cl}(\Gamma), \text{cl}(\Gamma^- \cap \psi^{-1}(\Gamma^+)) \right) < \delta \quad (18)$$

where dist stands for the Hausdorff distance.

- The previous definitions, in their full generality, will apply to cylinders which satisfy Condition (PS1), in the sense that one immediately shows that any such cylinder admits a homoclinic correspondence. When a cylinder \mathcal{C} satisfies (FS1), it turns out that the situation is much simpler: there exists a single C^1 diffeomorphism $\psi : \mathbf{A} \hookrightarrow$ which satisfy the previous compatibility condition with a single diffeomorphism $S : \text{Dom } S \rightarrow \text{Im } S$. In the case of a singular cylinder \mathcal{C}_\bullet with generalized section $\Sigma_\bullet \sim \mathbf{A}_\bullet$, there also exist a single C^1 diffeomorphism $\psi_\bullet : \mathbf{A}_\bullet \hookrightarrow$ which satisfy the previous compatibility condition with a single diffeomorphism $S : \text{Dom } S \rightarrow \text{Im } S$. The diffeomorphism ψ_\bullet is continuable to a diffeomorphism of ψ the continuation \mathbf{A} of the section.

3. Splitting arcs. The notions we introduce now are useful only in the case of cylinders satisfying (PS1). We consider a such a normally hyperbolic cylinder \mathcal{C} equipped with a twist section $\Sigma = j_\Sigma(\mathbf{A})$ and a homoclinic correspondence $\psi = (\psi_i)_{i \in I}$ on \mathbf{A} . An *arc* of \mathbf{A} is a continuous map $\zeta = [0, 1] \rightarrow \mathbf{A}$. We write $\hat{\zeta} = \zeta([0, 1]) \subset \mathbf{A}$ for the image of the arc. Given two distinct points θ, θ' of \mathbb{T} , we write $[\theta, \theta']$ for the (unique) segment bounded by θ and θ' according to the natural orientation of \mathbb{T} . When two points $\alpha = (\theta, r), \alpha' = (\theta', r')$ belong to a circle Γ which is a graph over \mathbb{T} , we write $[\alpha, \alpha']_\Gamma$ for the oriented segment of Γ located over $[\theta, \theta']$, equipped with the natural orientation of Γ . We write $-[\alpha, \alpha']_\Gamma$ for the segment equipped with the opposite orientation.

Consider $\Gamma \in \mathbf{Ess}(\varphi)$ and let $\alpha \in \Gamma$.

- A *splitting arc* based at α for the pair (φ, ψ) is an arc ζ of \mathbf{A} for which

$$\zeta(0) = \alpha, \quad \zeta([0, 1]) \subset \Gamma^-; \quad \exists i \in I, \quad \zeta([0, 1]) \subset \text{Dom } \psi_i, \quad \psi_i(\zeta([0, 1])) \subset \Gamma.$$

- A *splitting domain* based at α for the pair (φ, ψ) is a the interior of a 2-dimensional submanifold with boundary of \mathbf{A} which is contained in Γ^- and whose boundary contains a splitting arc based at α ;
- A *simple splitting arc* based at $\alpha = (\theta, r)$ for the pair (φ, ψ) is a splitting arc ζ based at α such that ζ projects over an interval $[\theta, \theta + \sigma]$ or an interval $[\theta - \sigma, \theta]$, with $0 < \sigma < \frac{1}{2}$.

4. Good cylinders. We will distinguish between three cases.

1) We say that a normally hyperbolic cylinder \mathcal{C} which satisfies (FS1) is a good cylinder when it admits a twist section $\Sigma = j_\Sigma(\mathbf{A})$ with return map $\varphi : \mathbf{A} \hookrightarrow$ and a homoclinic map $\psi : \mathbf{A} \hookrightarrow$ such that no element of $\mathbf{Ess}(\varphi)$ is invariant under ψ .

2) We say that a normally hyperbolic singular cylinder \mathcal{C}_\bullet which satisfies (FS1) is a good cylinder when it admits a generalized twist section $\Sigma_\bullet = j_\Sigma(\mathbf{A}_\bullet)$ with return map $\varphi : \mathbf{A}_\bullet \hookrightarrow$ (continuable as a twist map of an annulus) and a homoclinic map $\psi : \mathbf{A}_\bullet \hookrightarrow$ such that no element of $\mathbf{Ess}(\varphi)$ is invariant under ψ .

3) We say that a normally hyperbolic cylinder \mathcal{C} which satisfies (PS1) is a good cylinder when it admits a twist section $\Sigma = j_\Sigma(\mathbf{A})$ with return map $\varphi : \mathbf{A} \hookrightarrow$ and a homoclinic correspondence $\psi : \mathbf{A} \hookrightarrow$ such that

- for any element Γ of $\mathbf{Ess}(\varphi)$ which is not the upper boundary of a Birkhoff zone, there exists a splitting domain based on Γ ;
- if $\Gamma \in \mathbf{Ess}(\varphi)$ is the upper boundary of a Birkhoff zone, then there exists a simple splitting arc based on Γ .

5. Heteroclinic maps. Let \mathcal{C}_1 and \mathcal{C}_2 be disjoint good cylinders at energy \mathbf{e} for H , with characteristic projections $\Pi_i^\pm : W^\pm(\mathcal{C}_i) \rightarrow \mathcal{C}_i$ and twist sections $\Sigma_i = j_{\Sigma_i}(\mathbf{A}_i)$, with $\mathbf{A}_i = \mathbb{T} \times [a_i, b_i]$. We define the *transverse heteroclinic intersection of \mathcal{C}_1 and \mathcal{C}_2* as the set

$$\text{Hett}(\mathcal{C}_1, \mathcal{C}_2) \subset W^-(\mathcal{C}_1) \cap W^+(\mathcal{C}_2) \quad (19)$$

formed by the points ξ such that

$$W^-(\Pi_1^-(\xi)) \cap_\xi W^+(\mathcal{C}_2) \quad \text{and} \quad W^+(\Pi_2^+(\xi)) \cap_\xi W^-(\mathcal{C}_1). \quad (20)$$

A *heteroclinic map* from \mathcal{C}_1 to \mathcal{C}_2 is a C^1 diffeomorphism

$$\psi_1^2 : \text{Dom } \psi_1^2 \subset \Sigma_1 \rightarrow \text{Im } \psi_1^2 \subset \Sigma_2 \quad (21)$$

where $\text{Dom } \psi_1^2$ is an open neighborhood of $\partial^\bullet \Sigma_1$ in Σ_1 and $\text{Im } \psi_1^2$ is an open neighborhood of $\partial_\bullet \Sigma_2$ in \mathbf{A}_2 , for which there exists a C^1 diffeomorphism

$$S_1^2 : \text{Dom } S_1^2 \subset \mathcal{C}_1 \rightarrow \text{Im } S_1^2 \subset \mathcal{C}_2 \quad (22)$$

where $\text{Dom } S_1^2$ and $\text{Im } S_1^2$ are open subsets, which satisfies the following conditions:

- there exists a C^1 function $\tau : \text{Dom } \psi_1^2 \rightarrow \mathbb{R}$ such that

$$\forall x \in \text{Dom } \psi_1^2, \quad \Phi_H^{\tau(x)}(x) \in \text{Dom } S_1^2 \quad \text{and} \quad \psi_1^2(x) = S_1^2\left(\Phi_H^{\tau(x)}(x)\right); \quad (23)$$

- there is an open subset $S_1^2 \subset \text{Dom } S_1^2$, with full measure in $\text{Dom } S_1^2$, such that

$$\forall y \in S_1^2, \quad W^-(y) \cap W^+(S_1^2(y)) \cap \text{Hett}(\mathcal{C}_1, \mathcal{C}_2) \neq \emptyset. \quad (24)$$

6. Bifurcation maps. Let \mathcal{C}_1 and \mathcal{C}_2 be disjoint good cylinders at energy \mathbf{e} for H , with characteristic projections $\Pi_i^\pm : W^\pm(\mathcal{C}_i) \rightarrow \mathcal{C}_i$ and twist sections $\Sigma_i = j_{\Sigma_i}(\mathbf{A}_i)$, with $\mathbf{A}_i = \mathbb{T} \times [a_i, b_i]$. A *bifurcation map* from \mathcal{C}_1 to \mathcal{C}_2 is a C^1 diffeomorphism

$$\psi_1^2 : \Gamma_1 \subset \Sigma_1 \rightarrow \Gamma_2 \subset \Sigma_2 \quad (25)$$

where Γ_i are dynamically minimal circles for the return maps φ_i , such that, denoting by \mathcal{T}_i the essential tori they generate, there exists a C^1 diffeomorphism

$$S_1^2 : \mathcal{T}_1 \rightarrow \mathcal{T}_2 \quad (26)$$

which satisfies the following conditions: there exists a C^1 function $\tau : \mathcal{T}_1 \rightarrow \mathbb{R}$ such that

$$\forall x \in \mathcal{T}_1, \quad \Phi_H^{\tau(x)}(x) \in \mathcal{T}_1 \quad \text{and} \quad \psi_1^2(x) = S_1^2\left(\Phi_H^{\tau(x)}(x)\right); \quad (27)$$

and

$$\forall y \in \mathcal{T}_1, \quad W^-(y) \cap W^+(S_1^2(y)) \cap \text{Hett}(\mathcal{C}_1, \mathcal{C}_2) \neq \emptyset. \quad (28)$$

7. Good chains. A *good chain of cylinders* at energy \mathbf{e} is an admissible chain $(\mathcal{C}_k)_{1 \leq k \leq k_*}$ of *good* cylinders or singular cylinders at energy \mathbf{e} , with twist sections Σ_k , such that for $1 \leq k \leq k_* - 1$:

- either \mathcal{C}_k and \mathcal{C}_{k+1} are consecutive cylinders contained in the same cylinder, that is $\partial_\bullet \mathcal{C}_k = \partial^\bullet \mathcal{C}_{k+1}$, which satisfy the gluing condition (G);
- or there exists a bifurcation map ψ_k^{k+1} from \mathcal{C}_k to \mathcal{C}_{k+1} ;
- or a heteroclinic map ψ_k^{k+1} from \mathcal{C}_k to \mathcal{C}_{k+1} and a circle $\Gamma_k \in \mathbf{Ess}(\varphi_k)$ contained in $\text{Dom } \psi_k^{k+1}$ whose image $\psi_k^{k+1}(\Gamma_k)$ is a dynamically minimal essential invariant circle for φ_{k+1} .

3 Perturbation of characteristic foliations

We now introduce the main ingredient of our perturbative construction.

3.1 A perturbative lemma for Poincaré maps

We refer to [MS] for the necessary definitions and results in symplectic geometry. We begin with a global form of the Hamiltonian flow-box theorem.

Lemma 3.1. *Let (M^{2m}, Ω) be a symplectic manifold with Poisson bracket $\{, \}$, and fix a Hamiltonian $H \in C^\infty(M)$, with complete vector field. Let Λ be a codimension 1 submanifold of M , transverse to X_H , such that there exists an open interval $I \subset \mathbb{R}$ containing 0 for which the restriction of Φ_H to $I \times \Lambda$ is an embedding. Set*

$$\mathcal{D} := \Phi_H(I \times \Lambda) \quad (29)$$

and let $F : \mathcal{D} \rightarrow \mathbb{R}$ be the C^κ (transition time) function defined by

$$\Phi_H(-F(x), x) \in \Lambda, \quad \forall x \in \mathcal{D}. \quad (30)$$

Then $\{H, F\} = 1$ and $\Lambda = F^{-1}(0)$, so X_F is tangent to Λ . Assume moreover that there exist an open interval J and $\bar{\mathbf{e}} \in J$ such that, setting $\Lambda_{\bar{\mathbf{e}}} = H^{-1}(\bar{\mathbf{e}}) \cap \Lambda$, the flow of X_F is defined on $J \times \Lambda_{\bar{\mathbf{e}}}$ and satisfies

$$\Lambda = \Phi_F(J \times \Lambda_{\bar{\mathbf{e}}}). \quad (31)$$

Then the form $\Omega_{\bar{\mathbf{e}}}$ induced by Ω on $\Lambda_{\bar{\mathbf{e}}}$ is symplectic, and the map

$$\begin{aligned} \chi : (I \times J) \times \Lambda_{\bar{\mathbf{e}}} &\longrightarrow \mathcal{D} \\ ((t, \mathbf{e}), x) &\longmapsto \Phi_H(t, \Phi_F(\mathbf{e}, x)) \end{aligned} \quad (32)$$

is a C^∞ symplectic diffeomorphism on its image, where $(I \times J) \times \Lambda_{\bar{\mathbf{e}}}$ is equipped with the form

$$d\mathbf{e} \wedge dt \oplus \Omega_{\bar{\mathbf{e}}}. \quad (33)$$

Moreover

$$H \circ \chi((t, \mathbf{e}), x) = \mathbf{e}, \quad \chi^*(X_H) = \frac{\partial}{\partial t}. \quad (34)$$

In the following we say that a submanifold Λ satisfying the assumptions of the previous lemma is a *box-section* for X_H , with associated data $(I, J, \bar{\mathbf{e}})$. Given *any* transverse section $\hat{\Lambda}$ of X_H and a compact subset K of $\hat{\Lambda} \cap H^{-1}(\bar{\mathbf{e}})$, one easily proves that there exists a box-section $\Lambda \subset \hat{\Lambda}$ which is a neighborhood of K in $\hat{\Lambda}$.

Lemma 3.2. *Let M^{2m} be a symplectic manifold and fix a Hamiltonian $H \in C^\infty(M)$ with complete vector field X_H and flow Φ_H . Assume that Λ is a box-section for X_H , with associated data $(I, J, \bar{\mathbf{e}})$ such that $[-1, 0] \subset I$, set*

$$\Delta^{(-1)} = \Phi_H^{-1}(\Lambda), \quad \Lambda_{\bar{\mathbf{e}}} = \Lambda \cap H^{-1}(\bar{\mathbf{e}}), \quad \Delta_{\bar{\mathbf{e}}}^{(-1)} = \Delta^{(-1)} \cap H^{-1}(\bar{\mathbf{e}}), \quad (35)$$

and let P_H be the Φ_H -induced Poincaré map between $\Delta_{\bar{\mathbf{e}}}^{(-1)}$ and $\Lambda_{\bar{\mathbf{e}}}$.

Let K be a compact of $\Lambda_{\bar{\mathbf{e}}}$ contained in the relative interior of $\Lambda_{\bar{\mathbf{e}}}$. Then for any C^∞ Hamiltonian diffeomorphism $\phi : \Lambda_{\bar{\mathbf{e}}} \hookrightarrow$ with support in K , there exists a Hamiltonian $\mathcal{H} \in C^\infty(M)$ such that:

- Λ is a box-section of $X_{\mathcal{H}}$ with associated data $(I, J, \bar{\mathbf{e}})$, and $\Phi_{\mathcal{H}}^{-1}(\Lambda) = \Delta^{(-1)}$,
- \mathcal{H} coincides with H outside $\Phi_H([-1, 0] \times \Lambda)$,
- the $\Phi_{\mathcal{H}}$ -induced Poincaré map between $\Delta_{\bar{\mathbf{e}}}^{(-1)}$ and $\Lambda_{\bar{\mathbf{e}}}$ satisfies

$$P_{\mathcal{H}} = \phi \circ P_H. \quad (36)$$

- \mathcal{H} tends to H in the C^∞ topology when $\phi \rightarrow \text{Id}$ in the C^∞ topology.

Proof. The map

$$\chi : I \times J \times \Lambda_{\mathbf{e}} \longrightarrow \Phi_H(I \times \Lambda) \quad (37)$$

of Lemma 3.1 is a symplectic diffeomorphism such that $H \circ \chi(t, \mathbf{e}, x) = \mathbf{e}$. We first work in the coordinates (t, \mathbf{e}, x) to construct our new Hamiltonian. Let $\ell : I \times \Lambda_{\mathbf{e}}$ be a C^∞ nonautonomous Hamiltonian on $\Lambda_{\mathbf{e}}$ with support in $[-2/3, -1/3] \times K$, whose associated transition map between the times $\{-2/3\}$ and $\{-1/3\}$ coincides with ϕ . We set

$$\mathbf{H}(t, \mathbf{e}, x) = \mathbf{e} + \eta(\mathbf{e})\ell(t, x) \quad (38)$$

where $\eta : J \rightarrow \mathbb{R}$ is a C^∞ function with support in J , equal to 1 in an open neighborhood J_* of \mathbf{e} , so that the function $\eta\ell$ has compact support contained in $[-2/3, -1/3] \times I \times K$. The associated vector field reads, for $(t, \mathbf{e}, x) \in I \times J_* \times \Lambda_{\mathbf{e}}$:

$$\dot{t} = 1, \quad \dot{\mathbf{e}} = \partial_t \ell(t, x), \quad \dot{x} = X_\ell(t, x), \quad (39)$$

where X_ℓ stands for the vector field generated by ℓ relatively to the induced form $\Omega_{\bar{\mathbf{e}}}$ on $\Lambda_{\mathbf{e}}$. As a consequence, the Poincaré map of \mathbf{H} between the sections $\{-1\} \times (J \times \Lambda_{\mathbf{e}})$ and $\{0\} \times (J \times \Lambda_{\mathbf{e}})$ reads:

$$P_{\mathbf{H}}(-1, \mathbf{e}, x) = (0, \mathbf{e}, \phi(x)). \quad (40)$$

The function $\mathbf{H} \circ \chi^{-1}$ coincides with H in the open set

$$\chi(I \times J \times \Lambda_{\mathbf{e}}) \setminus \chi([-2/3, -1/3] \times \text{Supp } \eta \times K) \quad (41)$$

and so continues as a C^∞ function \mathcal{H} on M , which coincides with H outside the latter factor. Since χ is symplectic, Λ is a box-section of $X_{\mathcal{H}}$ with associated data $(I, J, \bar{\mathbf{e}})$, and $\Phi_{\mathcal{H}}^{-1}(\Lambda) = \Delta^{(-1)}$. The Poincaré maps $P_{\mathbf{H}}$ and $P_{\mathcal{H}}$ satisfy

$$P_{\mathcal{H}} = \chi \circ P_{\mathbf{H}} \circ \chi_{|\Phi_H^{-1}(\Lambda_{\bar{\mathbf{e}}})}^{-1}, \quad (42)$$

and

$$\chi(0, \bar{\mathbf{e}}, x) = x, \quad \chi(-1, \bar{\mathbf{e}}, x) = \Phi_H^{-1}(x), \quad (43)$$

hence, setting $z = \Phi_H^{-1}(x)$ for $x \in \Lambda_{\bar{\mathbf{e}}}$:

$$P_{\mathcal{H}}(z) = \chi \circ P_H \circ \chi^{-1}(z) = \chi \circ P_H(-1, \bar{\mathbf{e}}, x) = \chi(0, \mathbf{e}, \phi(x)) = \phi(x) \quad (44)$$

so that

$$P_{\mathcal{H}} = \phi \circ P_H. \quad (45)$$

Finally one can choose ℓ so that $\ell \rightarrow 0$ in the C^∞ topology when ϕ tends to Id in the C^∞ topology, from which our last assertion easily follows. \square

3.2 Perturbations of homoclinic maps in the (FS) case

In this section we consider a Hamiltonian $H \in C^\kappa(\mathbb{A}^3)$ and fix a regular value \mathbf{e} of H .

Lemma 3.3. *Assume that $\mathcal{C} \subset H^{-1}(\mathbf{e})$ with twist section Σ satisfies condition (FS), and let ψ_H be its homoclinic map. Then for any $C^{\kappa-2}$ Hamiltonian diffeomorphism $\sigma : \Sigma \hookrightarrow$, there exists a C^κ Hamiltonian \mathcal{H} which coincides with H in the neighborhood of \mathcal{C} , such that \mathcal{C} still satisfies condition (FS) for \mathcal{H} and that the associated homoclinic map $\psi_{\mathcal{H}}$ satisfies*

$$\psi_{\mathcal{H}} = \sigma \circ \psi_H; \quad (46)$$

moreover $\|\mathcal{H} - H\|_\kappa \rightarrow 0$ when $d_{\kappa-2}(\sigma, \text{Id}) \rightarrow 0$.

Proof. By the C^∞ -smoothing technique of [Zeh76], there exists a Hamiltonian H_* , arbitrarily close to H in the C^κ topology and coincides with H in the neighborhood of \mathcal{C} , such that \mathcal{C} still satisfies condition (FS) with respect to Δ and such that H_* is C^∞ in the neighborhood of Δ . Let ψ_{H_*} be the associated homoclinic map and set

$$\sigma_* = \sigma \circ \psi_H \circ \psi_{H_*}^{-1}, \quad (47)$$

so that σ_* is a Hamiltonian diffeomorphism of Σ , arbitrarily close to σ in the $C^{\kappa-2}$ topology. It is therefore enough to prove the result for H_* and σ_* instead of H and σ . So one can assume without loss of generality that H is C^∞ in the neighborhood of Δ .

By compactness of \mathcal{C}^- , one can find a neighborhood Λ of \mathcal{C}^- in Δ which is a box-section for X_H , with data (I, J, \mathbf{e}) , where I contains some interval $[-\tau, 0]$. One can moreover assume that H is C^∞ in the neighborhood of $\Phi_H(\bar{I} \times \bar{\Lambda})$.

Set

$$\phi = (P_H^+ \circ \Pi_H^+)^{-1} \circ \sigma \circ (P_H^+ \circ \Pi_H^+)|_{\mathcal{J}} \quad (48)$$

where $\mathcal{J} = \mathcal{C}^+ \cap \mathcal{C}^-$, so that ϕ is a Hamiltonian diffeomorphism of \mathcal{J} . We proved in [Mar16] the existence of a Hamiltonian diffeomorphism ϕ of $\Delta_{\mathbf{e}}$ which continues ϕ .

By Lemma 3.2, there exists a Hamiltonian $\mathcal{H} \in C^\kappa(M)$ such that:

- Λ is a box-section of $X_{\mathcal{H}}$ with associated data $(I, J, \bar{\mathbf{e}})$, and $\Phi_{\mathcal{H}}^{-1}(\Lambda) = \Delta^{(-1)}$,
- \mathcal{H} coincides with H outside $\Phi_H([-1, 0] \times \Lambda)$,
- the $\Phi_{\mathcal{H}}$ -induced Poincaré map between $\Lambda_{\bar{\mathbf{e}}}^-$ and $\Lambda_{\bar{\mathbf{e}}}$ satisfies

$$P_{\mathcal{H}} = \phi \circ P_H. \quad (49)$$

- \mathcal{H} tends to H in the C^κ topology when $\phi \rightarrow \text{Id}$ in the $C^{\kappa-2}$ topology.

Fix $\xi \in \mathcal{A}^-$ and set $\eta = \Phi_H^{-\tau}(\xi)$. Since H and \mathcal{H} coincide outside the perturbation box

$$\Pi_H^-(\eta) = \Pi_{\mathcal{H}}^-(\eta). \quad (50)$$

By equivariance of the unstable foliation of $W^-(\mathcal{A})$:

$$\Pi_{\mathcal{H}}^-(\Phi_{\mathcal{H}}(\eta)) = \Pi_H^-(\Phi_H(\eta)). \quad (51)$$

Moreover, if $\xi \in \mathcal{C}^-$,

$$\Phi_{\mathcal{H}}^{\tau}(\eta) = \phi \circ \Phi_H^{\tau}(\eta) \quad (52)$$

which proves that

$$\Pi_{\mathcal{H}}^- \circ \phi(\xi) = \Pi_H^-(\xi), \quad \forall \xi \in \mathcal{C}^-. \quad (53)$$

In particular, since ϕ leaves \mathcal{J} invariant

$$\Pi_{\mathcal{H}}^-(\mathcal{J}) = \Pi_H^-(\mathcal{J}). \quad (54)$$

and the Poincaré maps $P_{\mathcal{H}}^-$ and P_H^- coincide. Moreover, since the perturbation does not affect the stable manifold $W^+(\mathcal{A})$:

$$\psi_{\mathcal{H}}(x) = P_H^+ \circ \Pi_H^+ \circ j_{\mathcal{H}}^- \circ (P_H^-)^{-1}(x), \quad \forall x \in \Sigma. \quad (55)$$

Hence

$$\psi_{\mathcal{H}}(x) = P_H^+ \circ \Pi_H^+ \circ \phi \circ j_H^- \circ (P_H^-)^{-1}(x) = \sigma \circ \psi_H(x), \quad \forall x \in \Sigma, \quad (56)$$

which proves our claim. Finally, since $d_{\kappa-2}(\phi, \text{Id}) \rightarrow 0$ when $d_{\kappa-2}(\sigma, \text{Id}) \rightarrow 0$ (see [Mar16]), the last statement comes from Lemma 3.2. \square

The following lemma for the heteroclinic condition (FS) is proved exactly in the same way as the previous one.

Lemma 3.4. *Assume that the pair $(\mathcal{C}_0, \mathcal{C}_1)$ of compact invariant cylinders with twist sections Σ_i , contained in $H^{-1}(\mathbf{e})$, satisfies condition (FS), and let ψ_H be its heteroclinic map. Then for any $C^{\kappa-2}$ Hamiltonian diffeomorphism $\sigma : \Sigma_2 \hookrightarrow$, there exists a C^{κ} Hamiltonian \mathcal{H} which coincides with H in the neighborhood of \mathcal{C}_0 and \mathcal{C}_1 , such that $(\mathcal{C}_0, \mathcal{C}_1)$ still satisfies condition (FS) for \mathcal{H} and that the associated heteroclinic map $\psi_{\mathcal{H}}$ satisfies*

$$\psi_{\mathcal{H}} = \sigma \circ \psi_H; \quad (57)$$

moreover $\|\mathcal{H} - H\|_{\kappa} \rightarrow 0$ when $d_{\kappa-2}(\sigma, \text{Id}) \rightarrow 0$.

3.3 Perturbations of characteristic projections in the (PS) case

We consider a Hamiltonian $H \in C^{\kappa}(\mathbb{A}^3)$ and fix a regular value \mathbf{e} of H .

Lemma 3.5. *Assume that $\mathcal{C} \subset H^{-1}(\mathbf{e})$ with twist section Σ satisfies condition (PS). Then there exists a compact neighborhood K of \mathcal{C}^- in $\Delta_{\mathbf{e}}$ such that for any $C^{\kappa-2}$ Hamiltonian diffeomorphism ϕ of $\Delta_{\mathbf{e}}$ with support in K , there exists a C^{κ} Hamiltonian \mathcal{H} which coincides with H in the neighborhood of \mathcal{C} , such that \mathcal{C} still satisfies condition (PS) for \mathcal{H} and that the associated characteristic projection satisfies*

$$(\Pi_{\mathcal{H}}^-)_{|\mathcal{C}^-} = (\Pi_{\mathcal{H}}^-)_{|\mathcal{C}^-} \circ \phi; \quad (58)$$

moreover $\|\mathcal{H} - H\|_{\kappa} \rightarrow 0$ when $d_{\kappa-2}(\phi, \text{Id}) \rightarrow 0$.

Proof. The proof follows the same lines as that of Lemma 3.2. One can assume that H is C^∞ in the neighborhood of Δ . One first constructs a box-section $\Lambda \subset \Delta_{\mathbf{e}}$ with data (I, J, \mathbf{e}) , where I contains some interval $[-\tau, 0]$, such that moreover H is C^∞ in the neighborhood of $\Phi_H(\bar{I} \times \bar{\Lambda})$. Then, by Lemma 3.2, there exists a Hamiltonian $\mathcal{H} \in C^\kappa(M)$ such that:

- Λ is a box-section of $X_{\mathcal{H}}$ with associated data $(I, J, \bar{\mathbf{e}})$, and $\Phi_{\mathcal{H}}^{-1}(\Lambda) = \Delta^{(-1)}$,
- \mathcal{H} coincides with H outside $\Phi_H([-1, 0] \times \Lambda)$,
- the $\Phi_{\mathcal{H}}$ -induced Poincaré map between $\Lambda_{\bar{\mathbf{e}}}^-$ and $\Lambda_{\bar{\mathbf{e}}}^+$ satisfies

$$P_{\mathcal{H}} = \phi \circ P_H. \quad (59)$$

- \mathcal{H} tends to H in the C^κ topology when $\phi \rightarrow \text{Id}$ in the $C^{\kappa-2}$ topology.

One then deduces exactly as in Lemma 3.2 that

$$(\Pi_{\mathcal{H}}^-)_{|\mathcal{C}^-} = (\Pi_{\mathcal{H}}^-)_{|\mathcal{C}^-} \circ \phi; \quad (60)$$

and the last statement is immediate. \square

4 Proofs of Theorem 2 and Theorem 1

We now use the results of the previous section and prove that admissible chains can be made good chains by arbitrarily small perturbations of the Hamiltonian, which is the content of Theorem 2. We then prove Theorem 1, which relies on the results of [Mar, GM].

4.1 Cylinders with condition (FS)

We consider a proper Hamiltonian $H \in C^\kappa(\mathbb{A}^3)$ and fix a regular value \mathbf{e} of H .

Lemma 4.1. *Assume that $\mathcal{C} \subset H^{-1}(\mathbf{e})$ satisfies condition (FS1). Then for any $\alpha > 0$, there exists a C^κ Hamiltonian \mathcal{H} such that \mathcal{C} is a good cylinder for \mathcal{H} , and which satisfies*

$$\|\mathcal{H} - H\|_\kappa < \alpha. \quad (61)$$

Proof. By [R] applied to the symplectic manifold \mathcal{A} , there exists an arbitrarily small perturbation \tilde{H} of H in the C^κ topology which admits an invariant annulus $\tilde{\mathcal{A}}$ and an invariant cylinder $\tilde{\mathcal{C}}$ which are arbitrarily C^κ -close to the initial ones, such that any periodic orbit of \tilde{H} is hyperbolic in $\tilde{\mathcal{C}}$ or elliptic with nondegenerate torsion and KAM nonresonance conditions.

As a consequence, one can choose \tilde{H} such that $\|H - \tilde{H}\|_{C^\kappa} < \alpha/2$, $\tilde{\mathcal{C}}$ still satisfies condition (FS1) and admits a section Σ for which the Poincaré map $\tilde{\varphi}$ is a special twist map. We can therefore assume that these properties are satisfied by the initial Hamiltonian H and we get rid of the \sim .

Let ψ be the homoclinic map of \mathcal{C} for H . By [M02], there exists a C^κ Hamiltonian diffeomorphism σ of Σ , arbitrarily close to Id , such that the return map $\varphi : \Sigma \hookrightarrow \Sigma$ and the map $\sigma^{-1} \circ \psi \circ \sigma : \Sigma \hookrightarrow \Sigma$ have no common essential invariant circle. Observe that $[\sigma^{-1} \circ \psi \circ \sigma \circ \psi^{-1}]$ is a Hamiltonian diffeomorphism of Σ , which tends to Id in the C^κ topology when σ tends to Id in the C^κ topology. So, by Lemma 3.3, if σ is close enough to the identity, there exists a Hamiltonian \mathcal{H} which coincides with H in the neighborhood of \mathcal{C} , such that \mathcal{C} still satisfies condition (FS1), with new homoclinic map

$$\psi_{\mathcal{H}} = [\sigma^{-1} \circ \psi \circ \sigma \circ \psi^{-1}] \circ \psi = \sigma^{-1} \circ \psi \circ \sigma. \quad (62)$$

One can choose σ so that $\|\mathcal{H} - H\|_\kappa < \alpha$. By construction, the cylinder \mathcal{C} is now a good cylinder at energy \mathbf{e} for \mathcal{H} . \square

4.2 Cylinders with condition (PS)

We consider a proper Hamiltonian $H \in C^\kappa(\mathbb{A}^3)$ and fix a regular value \mathbf{e} of H . We prove the following analog of Lemma 4.1 for condition (PS1).

Lemma 4.2. *Assume that $\mathcal{C} \subset H^{-1}(\mathbf{e})$ satisfies condition (PS1). Then for any $\alpha > 0$, there exists a C^κ Hamiltonian \mathcal{H} such that \mathcal{C} is a good cylinder for \mathcal{H} , and which satisfies*

$$\|\mathcal{H} - H\|_\kappa < \alpha. \quad (63)$$

The proof follows from a sequence of intermediate lemmas. First, by the same argument as in the proof of the previous lemma, we can assume that $\mathcal{C} \subset H^{-1}(\mathbf{e})$ admits a section Σ whose attached return map φ is a special twist map. We set

$$\mathcal{C}_* = \mathcal{A} \cap H^{-1}(\mathbf{e}), \quad \mathcal{C}_*^\pm = j^\pm(\mathcal{C}_*) \quad (64)$$

so that $\mathcal{C} \subset \mathcal{C}_*$ and $\mathcal{C}^\pm \subset \mathcal{C}_*^\pm$.

1. The first lemma is a direct consequence of the last two conditions of (PS1), of which we keep the notation.

Lemma 4.3. *For each $\mathcal{T} \in \text{Tess}(\mathcal{C})$, $\mathcal{T}^+ \cap \mathcal{T}^- \neq \emptyset$.*

Proof. The Lipschitzian Lagrangian tori $\Psi^{sec}(\mathcal{T}^+)$ and $\Psi^{sec}(\mathcal{T}^-)$ are graphs over the null section in \mathcal{O}^{sec} . They moreover have the same cohomology, since they are images of the same graph $\Psi^{ann}(\mathcal{T})$ by the exact-symplectic diffeomorphisms

$$\Psi^{sec} \circ j^\pm \circ (\Psi^{ann})^{-1}.$$

Hence their intersection is nonempty. \square

2. Given a vector field X on \mathcal{C}_* and a 2-dimensional submanifold $S \subset \mathcal{C}_*$, we define the tangency set

$$\text{Tan}(X, S) = \{x \in S \mid X(x) \in T_x S\}.$$

We say that a point $x \in S \setminus \text{Tan}(X, S)$ is regular. We define the folds and cusps of \mathcal{Z} relatively to X in the usual way (see [GS]).

3. We fix a compact subset $K \subset \Delta_{\mathbf{e}}$ which contains $\mathcal{C}^+ \cup \mathcal{C}^-$ in its interior, and we denote by $\mathbf{H}(K)$ the space of pairs of nonautonomous Hamiltonians in $C^\kappa(\mathbb{R} \times \Delta_{\mathbf{e}})$ with support in $[-2/3, 1/3] \times K$. Let \mathbf{H}_0 be a ball centered at 0 in the space $\mathbf{H}(K)$, such that for each $(\ell_+, \ell_-) \in \mathbf{H}_0$, the conclusion of Lemma 3.5 holds. Given $(\ell_+, \ell_-) \in \mathbf{H}(K)$, we denote by $\mathcal{H}_{(\ell_+, \ell_-)}$ the associated Hamiltonian, as defined in Lemma 3.5. Recall that $\mathcal{H}_{(\ell_+, \ell_-)}$ coincides with the initial Hamiltonian in the neighborhood of \mathcal{A} . We can therefore assume that \mathbf{H}_0 is small enough so that \mathcal{C} still satisfies condition (PS1) for $\mathcal{H}_{(\ell_+, \ell_-)}$, relatively to the same section Δ , with characteristic maps

$$j_{(\ell_+, \ell_-)}^\pm : \mathcal{A} \rightarrow \mathcal{A}^\pm(\ell_+, \ell_-) \subset \Delta.$$

In order not to overload the notation and get rid of the problem of the boundaries and corners when considering intersections, we continue \mathcal{C} to a slightly larger 2-dimensional

manifold *without boundary* contained in \mathcal{C}_* and with compact closure, that we still denote by \mathcal{C} . We set

$$\mathcal{C}^\pm(\ell_+, \ell_-) = j_{(\ell_+, \ell_-)}^\pm(\mathcal{C}), \quad \mathcal{C}_*^\pm(\ell_+, \ell_-) = j_{(\ell_+, \ell_-)}^\pm(\mathcal{C}_*).$$

Lemma 4.4. *The following properties are satisfied.*

1. The set \mathbf{H}_1 of pairs of Hamiltonians $(\ell_+, \ell_-) \in \mathbf{H}_0$ for which the intersection

$$\mathcal{C}_*^+(\ell_+, \ell_-) \cap \mathcal{C}_*^-(\ell_+, \ell_-)$$

is transverse in $\Delta_{\mathbf{e}}$ at each point of $\text{cl}(\mathcal{C})$ is open and dense in \mathbf{H}_0 .

2. For $(\ell_+, \ell_-) \in \mathbf{H}_1$, the set

$$\mathcal{I}(\ell_+, \ell_-) := \mathcal{C}^+(\ell_+, \ell_-) \cap \mathcal{C}^-(\ell_+, \ell_-)$$

is a two-dimensional submanifold of $\Delta_{\mathbf{e}}$ which contains the set $\mathcal{X}(\ell_+, \ell_-)$ of all homoclinic intersections $j^+(\ell_+, \ell_-)(\mathcal{T}) \cap j^-(\ell_+, \ell_-)(\mathcal{T})$ for $\mathcal{T} \in \mathbf{Tess}(\mathcal{C})$.

3. We get rid of the indexation by (ℓ_+, ℓ_-) when obvious. The set \mathbf{H}_2 of pairs of Hamiltonians in \mathbf{H}_1 for which the subsets

$$\left\{x \in \mathcal{I} \mid X^-(x) \in T_x W^+(\mathcal{C})\right\}, \quad \left\{x \in \mathcal{I} \mid X^+(x) \in T_x W^-(\mathcal{C})\right\},$$

are 1-dimensional submanifolds of \mathcal{I}_* , is open dense in \mathbf{H}_0 .

4. The set \mathbf{H}_3 of pairs of Hamiltonians in \mathbf{H}_2 for which the tangency sets

$$\mathcal{Z}^\pm = \text{Tan}(X_{\mathcal{H}}, \Pi^\pm(\mathcal{I}_*))$$

are 1-dimensional submanifolds of $\Pi^\pm(\mathcal{I}_*)$, is open and dense in \mathbf{H}_0 .

5. The set \mathbf{H}_4 of pairs of Hamiltonians in \mathbf{H}_3 for which

$$\mathcal{X} \cap (\Pi^+)^{-1}(\mathcal{Z}^+) \cap (\Pi^-)^{-1}(\mathcal{Z}^-) = \emptyset \tag{65}$$

is open and dense in \mathbf{H}_0 .

6. The set \mathbf{H}_5 of pairs of Hamiltonians in \mathbf{H}_4 for which each point of $\Pi^+(\mathcal{X}) \cap \mathcal{Z}^+$ and $\Pi^-(\mathcal{X}) \cap \mathcal{Z}^-$ is either regular or located on a fold, is open and dense set in \mathbf{H}_0 .

Proof. The proofs use only very classical methods of singularity theory, we will give the main ideas and refer to [LM] for more details. We first recall the following classical genericity result by Abraham.

Theorem ([AR67]). *Fix $1 \leq k < +\infty$. Let \mathcal{L} be a C^k and second-countable Banach manifold. Let X and Y be finite dimensional C^k manifolds. Let $\chi : \mathcal{L} \rightarrow C^k(X, Y)$ be a map such that the associated evaluation*

$$\text{ev}_\chi : (\mathcal{L} \times X) \rightarrow Y, \quad \text{ev}_\chi(\ell, x) = (\chi(\ell))(x)$$

is C^k for the natural structures. Fix a be a submanifold D of Y such that

$$k > \dim X - \text{codim } D$$

and assume that \mathbf{ev}_χ is transverse to D . Then the set of all $\ell \in \mathcal{L}$ such that $\chi(\ell)$ is transverse to D is residual in \mathcal{L} , and open and dense when D is compact.

- To prove 1, we consider the following map

$$\left\{ \begin{array}{ll} \chi : \mathbf{H}_0 & \rightarrow C^\kappa(\mathcal{C}_* \times \mathcal{C}_*, \Delta_{\mathbf{e}} \times \Delta_{\mathbf{e}}) \\ (\ell_+, \ell_-) & \mapsto \left[(x, y) \mapsto (j_{(\ell_+, \ell_-)}^+(x), j_{(\ell_+, \ell_-)}^-(y)) \right] \end{array} \right. \quad (66)$$

whose evaluation \mathbf{ev}_χ map is $C^{\kappa-2}$. Introduce the diagonal

$$D = \{(z, z) \mid z \in \Delta_{\mathbf{e}}\} \subset \Delta_{\mathbf{e}} \times \Delta_{\mathbf{e}}. \quad (67)$$

Given $(\ell_+^0, \ell_-^0) \in \mathbf{H}_0$, a point $(x, y) \in \mathcal{C}_* \times \mathcal{C}_*$ such that $\mathbf{ev}_\chi((\ell_+^0, \ell_-^0), x, y) \in D$ and a vector $u \in T_y \Delta_{\mathbf{e}}$, one readily checks the existence of a path $s \mapsto (\ell_+, \ell_-)(s)$ such that

$$(\ell_+, \ell_-)(s) = (\ell_+^0, \ell_-^0), \quad \frac{d}{ds} \mathbf{ev}_\chi(\ell(s), x, y)|_{s=0} = (0, u), \quad (68)$$

which proves that \mathbf{ev}_χ is transverse to D at (ℓ, x, y) . For κ large enough, the Abraham genericity theorem proves our first claim, while the second one is an immediate consequence of Lemma 4.3.

- To prove 3, we endow \mathbb{A}^3 with a trivial Riemannian metric, and we denote by \cdot the associated scalar product, which is therefore defined for pairs of vectors possibly tangent at different points. Given $x \in \mathcal{A}^\pm$, we denote by $N^\pm(x)$ the (suitably oriented) unit normal vectors at x to the invariant manifolds $W^\pm(\mathcal{A})$. We then introduce the map (where we get rid of the obvious indexation of j^\pm by (ℓ_+, ℓ_-)):

$$\left\{ \begin{array}{ll} \chi : \mathbf{H}_0 & \rightarrow C^\kappa(\mathcal{C}_* \times \mathcal{C}_*, \Delta_{\mathbf{e}} \times \Delta_{\mathbf{e}} \times \mathbb{R}) \\ (\ell_+, \ell_-) & \mapsto \left[(x, y) \mapsto (j^+(x), j^-(y), X^-(j^-(x)) \cdot N^+(j^+(y))) \right] \end{array} \right. \quad (69)$$

whose evaluation \mathbf{ev}_χ map is $C^{\kappa-2}$. One then proves by straightforward constructions that \mathbf{ev}_χ is transverse to the diagonal

$$D = \{(\xi, \xi, 0) \mid \xi \in \Delta_{\mathbf{e}}\}.$$

As a consequence of the Abraham theorem, for an open dense subset of pairs (ℓ_+, ℓ_-) , $\hat{\chi} := \chi(\ell_+, \ell_-)$ is transverse to D . For such pairs, $\hat{\chi}^{-1}(D)$ is a codimension 5 submanifold of $\mathcal{C}_* \times \mathcal{C}_*$, and so is 1-dimensional. The projection of this manifold on the first factor is the set of points.

- The proof of 4 is analogous. We denote by Y^\pm the direct image of the vector field X_H restricted to \mathcal{A} by the transition maps j^\pm , which is therefore a vector field on \mathcal{A}^\pm . We introduce the projection operator P on the normal bundle of the intersection \mathcal{I} , that we see as a submanifold of the trivial bundle $T\mathbb{A}^3$. To deal with \mathcal{Z}^- , we introduce the following map

$$\left\{ \begin{array}{ll} \chi : \mathbf{H}_0 & \rightarrow C^\kappa(\mathcal{C}_* \times \mathcal{C}_*, \Delta_{\mathbf{e}} \times \Delta_{\mathbf{e}} \times \mathbb{R}) \\ (\ell_+, \ell_-) & \mapsto \left[(x, y) \mapsto (j^+(x), j^-(y), \|P_{j^+(x)}(Y^-(y))\|^2) \right] \end{array} \right. \quad (70)$$

which yields the same result as above.

- To prove 5 we begin by proving that the set of pairs $(\ell_+, \ell_-) \in \mathbf{H}_4$ for which

$$\mathcal{X} \cap (\Pi^+)^{-1}(\mathcal{Z}^+) \cap (\Pi^-)^{-1}(\mathcal{Z}^-)$$

is finite is open and dense in \mathbf{H}_0 . For this we introduce the map

$$\chi : \mathbf{H}_0 \rightarrow C^\kappa(\mathcal{C}_* \times \mathcal{C}_*, \Delta_{\mathbf{e}} \times \Delta_{\mathbf{e}} \times \mathbb{R} \times \mathbb{R})$$

such that

$$(\chi(\ell_+, \ell_-))(x, y) = \left(j^+(x), j^-(y), Y^-(j^-(x)) \cdot N^+(j^+(y)), Y^+(j^+(x)) \cdot N^-(j^-(y)) \right)$$

which is also easily proved to be transverse to the diagonal

$$D = \{(\xi, \xi, 0, 0) \mid \xi \in \Delta_{\mathbf{e}}\}.$$

So the set of pairs for which $\hat{\chi}^{-1}(D)$ is a dimension 0 manifold is open and dense, which proves our claim. It only remains to check that one can make an additional perturbation which disconnects \mathcal{X} from this last set, which is easy.

- The proof of 6 is analogous to the previous one, since the set of cusp points is finite. \square

4. The following lemma immediately yields Lemma 4.2. Recall that \mathcal{C} admits a section Σ for which the return map φ is a special twist map.

Lemma 4.5. *Given $(\ell_+^0, \ell_-^0) \in \mathbf{H}_5$, with associated Hamiltonian \mathcal{H}^0 , there exists \mathcal{H} in $C^\kappa(\mathbb{A}^3)$, arbitrarily close to \mathcal{H}^0 in the C^κ topology and which coincides with \mathcal{H}^0 in the neighborhood of \mathcal{A} , for which:*

- \mathcal{C} satisfies condition (PS1) and admits a homoclinic correspondence ψ ;
- for any essential circle $\Gamma \in \mathbf{Ess}(\varphi)$ there exists a splitting arc based on Γ for (φ, ψ) ;
- if moreover Γ is the lower boundary of a Birkhoff zone, there exists a simple arc based of Γ for (φ, ψ) .

Proof. We will first prove that for any $\Gamma \in \mathbf{Ess}(\varphi^0)$ there exists a splitting arc based on Γ for (φ^0, ψ^0) . Let $\mathcal{T} \subset \mathcal{C}$ be the torus generated by Γ under the action of the Hamiltonian flow. The complement $\mathcal{C} \setminus \mathcal{T}$ admits two connected components, which we denote by \mathcal{C}_\bullet and \mathcal{C}^\bullet according to the orientation of Σ induced by its parametrization.

- Set $I = \mathcal{T}^+ \cap \mathcal{T}^-$, so that I is compact and contained in \mathcal{X} . By property 5) of Lemma 4.4, for each point $\xi \in I$, one of the points $\xi^\pm = \Pi^\pm(\xi)$ is regular with respect to X_H , in the sense that X_H is not tangent to $\Pi^\pm(\mathcal{T})$ at ξ^\pm . We say that ξ is positively (resp. negatively) regular when ξ^+ (resp. ξ^-) is regular.

We define the point x^+ as the first intersection of the positive orbit of ξ^+ with Σ , and the point x^- as the first intersection of the negative orbit of ξ^- with Σ . We adopt the same convention for the transport to Σ for all points in small enough neighborhoods of the previous two ones.

- Assume for instance that ξ is positively regular. Then there exists a neighborhood $V(\xi)$ of ξ in \mathcal{T} such that the previous transport by the Hamiltonian flow in \mathcal{C} induces an

embedding of the image $\Pi^\pm(V(\xi))$ in the section Σ , whose image V^+ is a neighborhood of the point x^+ . Hence one can moreover assume $V(\xi)$ small enough so that the complement $\hat{\Gamma} = V^+ \setminus \Gamma$ admits exactly two connected components, contained in \mathcal{C}^\bullet and \mathcal{C}_\bullet . Inverse transport of $\hat{\Gamma}$ by the Hamiltonian flow and then by j^+ yields a Lipschitzian curve in $V(\xi)$, which disconnects $V(\xi)$. We denote by $V^\bullet(\xi)$ and $V_\bullet(\xi)$ the two components of its complement, according to the intersections of their images with \mathcal{C}_\bullet and \mathcal{C}^\bullet . We say that \mathcal{C}^\bullet is the positive component and that \mathcal{C}_\bullet is the negative component.

- Then, the point ξ^- is either regular or located on a fold of $\Pi^-(\mathcal{J})$ relatively to X_H . One can reduce $V(\xi)$ in order that the intersection $\Pi^-(V(\xi)) \cap \mathcal{J}$ is connected, and therefore arc-connected. We will assume this condition satisfied in the following. This yields by inverse transport an arc-connected subset $\sigma = j^-(\sigma^-) \subset V(\xi)$. We say that the intersection I is *one-sided in $V(\xi)$* if either $\sigma \cap V_\bullet(\xi) = \emptyset$ or $\sigma \cap V^\bullet(\xi) = \emptyset$ is empty, we say that it is positive in the latter case and negative in the former.

- In the case where ξ is negatively regular, define the neighborhood $V(\xi)$ in a symmetric way, with the analogues for the notion of one-sided, positive and negative intersections in $V(\xi)$. When both ξ^+ and ξ^- are regular, we arbitrarily choose one of them to perform the previous construction and define the notion of one-sided intersection.

- One then gets a covering of I by a finite number of neighborhoods $V(\xi_1), \dots, V(\xi_\ell)$ with the previous properties. Arguing by contraction and assuming that each positively regular point ξ_i yields a positive intersection and each negatively regular point yields a negative intersection, one proves that it would be possible to construct a pair of arbitrarily close to the identity Hamiltonian diffeomorphisms ϕ^\pm of Δ_e to which Lemma 3.5 applies, and which yield a C^κ perturbation \mathcal{H} of H which still admits \mathcal{C} as a cylinder satisfying (PS1), and \mathcal{J} as an invariant torus, but for which $\mathcal{J}^+ \cap \mathcal{J}^-$ is empty, which is impossible.

- Therefore there exists in I either a positively regular point with two-sided or negative intersection, or a negatively regular point with two-sided or positive intersection. Both cases yield a splitting arc based on Γ for the (local) composition $P^+ \circ \Pi^+ \circ j - \circ (P^-)^{-1}$. Now we know that if for instance ξ is positively regular, then ξ^- is either regular or located on a fold. This immediately yields a splitting domain for the homoclinic correspondence and concludes the proof. \square

5. It only remains now to consider the case of lower boundaries of Birkhoff zones.

Lemma 4.6. *If Γ is the lower boundary of a Birkhoff zone, there exists a splitting arc based on Γ .*

Proof. The main observation is that the set of essential circles which are lower boundaries of Birkhoff zones is countable. Therefore it is enough to prove the property for one such circle. But this is an immediate perturbation of the previous lemma, by construction a pair of Hamiltonian diffeomorphisms which makes the intersection point of the previous splitting arc be a derivability point for the arc, with a non-vertical tangent, and a derivability point for the circle, with a tangent distinct from the tangent to the arc. \square

4.3 From admissible chains to good chains: proof of Theorem 2

Theorem 2 is now a direct consequence of the previous two sections. Consider an admissible chain $(\mathcal{C}_k)_{1 \leq k \leq k_*}$ with δ -bounded homoclinic correspondence at energy e for H . Then by

Lemma 4.1 used recursively there exists a Hamiltonian \mathcal{H}^0 arbitrarily C^κ -close to H such that each cylinder satisfying (FS1) is a good cylinder, and such that each gluing boundary still admits transverse homoclinic intersections. By Lemma 4.2 used recursively, one can then find \mathcal{H}^1 arbitrarily C^κ -close to \mathcal{H}^0 such that each cylinder (PS1) is a good cylinder, without altering the heteroclinic condition (PS2) or the gluing condition (G).

The only remaining point to prove is that one perturb the system in such a way that each bifurcation pair $(\mathcal{C}_k, \mathcal{C}_{k+1})$ admits a bifurcation map in the sense of Section 2.3, paragraph 6. We keep the assumptions and notation of [Mar], Part I, Section 2. We can first make family of small C^κ perturbations to the Hamiltonian $H_\varepsilon = h + \varepsilon f$, parametrized by ε , to make it C^∞ in the neighborhood of $\mathbb{T}^3 \times \{b\}$. We so construct a family of C^∞ Hamiltonians \mathcal{H}_ε such that $\|\tilde{H}_\varepsilon - H_\varepsilon\|_{C^p} \leq \alpha\varepsilon$, which we still write

$$\tilde{H}_\varepsilon = \tilde{h}_\varepsilon + \varepsilon \tilde{f}_\varepsilon = h + \varepsilon \hat{f}_\varepsilon,$$

so that \hat{f}_ε is C^p α -close to f . We then perturb \tilde{f}_ε in order that its bifurcation point admits a frequency vector ω which is 2-Diophantine, that is, in adapted coordinates: $\omega = (\hat{\omega}, 0)$ with $\hat{\omega}$ Diophantine. One then use the ε -dependent normal form of \mathbf{H}_ε introduced in [Mar], Appendix 3. Given two constants $d > 0$ and $\delta < 1$ with $1 - \delta > d$, then, there is an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, there exists an analytic symplectic embedding

$$\Phi_\varepsilon : \mathbb{T}^3 \times B(b_1, \varepsilon^d) \rightarrow \mathbb{T}^3 \times B(b_1, 2\varepsilon^d)$$

such that

$$\tilde{N}_\varepsilon(\theta, r) = \tilde{H}_\varepsilon \circ \Phi_\varepsilon(\theta, r) = \tilde{h}_\varepsilon(r) + g_\varepsilon(\theta_3, r) + R_\varepsilon(\theta, r),$$

where g_ε and R_ε are C^p functions and

$$\|R_\varepsilon\|_{C^p(\mathbb{T}^n \times B(b_1, \varepsilon^d))} \leq \varepsilon^3. \quad (71)$$

Moreover, Φ_ε is close to the identity, in the sense that

$$\|\Phi_\varepsilon - \text{Id}\|_{C^p(\mathbb{T}^3 \times B(b_1, \varepsilon^d))} \leq \varepsilon^{1-\delta}. \quad (72)$$

The final perturbation of the initial Hamiltonian (localized in the neighborhood of the bifurcation locus) will be the truncation

$$H_\varepsilon^0 = (\tilde{h}_\varepsilon + g_\varepsilon) \circ \Phi_\varepsilon^{-1} = h + \varepsilon f_\varepsilon^0$$

which is α close to H_ε when ε is small enough. One checks that H_ε^0 satisfies the condition of a good chain for the cylinders at their bifurcation point and that f_ε^0 is α close to f if ε is small enough. This concludes our proof.

4.4 Proof of Theorem 1

Fix $\mathbf{e} > \text{Min } h$ together with a finite family of arbitrary open sets O_1, \dots, O_m which intersect $h^{-1}(\mathbf{e})$. By [Mar], for $\kappa \geq \kappa_0$ large enough, there exists a lower-semicontinuous function

$$\varepsilon_0 : \mathcal{S}^\kappa \rightarrow \mathbb{R}^+$$

with positive values on an open dense subset of \mathcal{S}^κ such that for $f \in \mathcal{B}^\kappa(\varepsilon_0)$ the system

$$H(\theta, r) = h(r) + f(\theta, r) \quad (73)$$

admits an admissible chain $(\mathcal{C}_k)_{1 \leq k \leq k_*}$, with $(\delta/2)$ -bounded homoclinic correspondence, such that each open set $\mathbb{T}^3 \times O_k$ contains the δ -neighborhood in \mathbb{A}^3 of some essential torus of the chain. By Theorem 2 there exists \mathcal{H} , arbitrarily C^κ -close to H , such that $(\mathcal{C}_k)_{1 \leq k \leq k_*}$ is a good chain with δ -bounded homoclinic correspondence. By [GM] there exists an orbit for \mathcal{H} which is δ -admissible, and therefore which intersects each $\mathbb{T}^3 \times O_i$. This last property being open, Theorem 1 is proved.

A Normal hyperbolicity and symplectic geometry

We refer to [Ber10a, BB13, Cha04, Cha08, HPS77] for the references on normal hyperbolicity. Here we limit ourselves to a very simple class of systems which admit a normally hyperbolic invariant (non compact) submanifold, which serves us as a model from which all other definitions and properties will be deduced.

1. The following statement is a simple version of the persistence theorem for normally hyperbolic manifolds well-adapted to our setting, whose germ can be found in [Ber10b] and whose proof can be deduced from the previous references.

The normally hyperbolic persistence theorem. *Fix $m \geq 1$ and consider a vector field on \mathbb{R}^{m+2} of the form $\mathcal{V} = \mathcal{V}_0 + \mathcal{F}$, with \mathcal{V}_0 and \mathcal{F} of class C^1 and reads*

$$\dot{x} = X(x, u, s), \quad \dot{u} = \lambda_u(x) u, \quad \dot{s} = -\lambda_s(x) s, \quad (74)$$

for $(x, u, s) \in \mathbb{R}^{m+2}$. Assume moreover that there exists $\lambda > 0$ such that the inequalities

$$\lambda_u(x) \geq \lambda, \quad \text{and} \quad \lambda_s(x) \geq \lambda, \quad x \in \mathbb{R}^m. \quad (75)$$

hold. Fix a constant $\mu > 0$. Then there exists a constant $\delta_* > 0$ such that if

$$\|\partial_x X\|_{C^0(\mathbb{R}^{m+2})} \leq \delta_*, \quad \|\mathcal{F}\|_{C^1(\mathbb{R}^{m+2})} \leq \delta_*, \quad (76)$$

the following assertions hold.

- The maximal invariant set for \mathcal{V} contained in $O = \{(x, u, s) \in \mathbb{R}^{m+2} \mid \|(u, s)\| \leq \mu\}$ is an m -dimensional manifold $\text{Ann}(\mathcal{V})$ which admits the graph representation:

$$\text{Ann}(\mathcal{V}) = \{(x, u = U(x), s = S(x)) \mid x \in \mathbb{R}^m\},$$

where U and S are C^1 maps $\mathbb{R}^m \rightarrow \mathbb{R}$ such that

$$\|(U, S)\|_{C^0(\mathbb{R}^m)} \leq \frac{2}{\lambda} \|\mathcal{F}\|_{C^0}. \quad (77)$$

- The maximal positively invariant set for \mathcal{V} contained in O is an $(m+1)$ -dimensional manifold $W^+(\text{Ann}(\mathcal{V}))$ which admits the graph representation:

$$W^+(\text{Ann}(\mathcal{V})) = \{(x, u = U^+(x, s), s) \mid x \in \mathbb{R}^m, s \in [-\mu, \mu]\},$$

where U^+ is a C^1 map $\mathbb{R}^m \times [-1, 1] \rightarrow \mathbb{R}$ such that

$$\|U^+\|_{C^0(\mathbb{R}^m)} \leq c_+ \|\mathcal{F}\|_{C^0}. \quad (78)$$

for a suitable $c_+ > 0$. Moreover, there exists $C > 0$ such that for $w \in W^+(\text{Ann}(\mathcal{V}))$,

$$\text{dist}(\Phi^t(w), \text{Ann}(\mathcal{V})) \leq C \exp(-\lambda t), \quad t \geq 0. \quad (79)$$

- The maximal negatively invariant set for \mathcal{V} contained in O is an $(m+1)$ -dimensional manifold $W^-(\text{Ann}(\mathcal{V}))$ which admits the graph representation:

$$W^-(\text{Ann}(\mathcal{V})) = \{(x, u, s = S^-(x, u)) \mid x \in \mathbb{R}^m, u \in [-\mu, \mu]\},$$

where S^- is a C^1 map $\mathbb{R}^m \times [-1, 1] \rightarrow \mathbb{R}$ such that

$$\|S^-\|_{C^0(\mathbb{R}^m)} \leq c_- \|\mathcal{F}\|_{C^0}. \quad (80)$$

for a suitable $c_- > 0$. Moreover, there exists $C > 0$ such that for $w \in W^-(\text{Ann}(\mathcal{V}))$,

$$\text{dist}(\Phi^t(w), \text{Ann}(\mathcal{V})) \leq C \exp(\lambda t), \quad t \leq 0. \quad (81)$$

- The manifolds $W^\pm(\text{Ann}(\mathcal{V}))$ admit C^0 foliations $(W^\pm(x))_{x \in \text{Ann}(\mathcal{V})}$ such that for $w \in W^\pm(x)$

$$\text{dist}(\Phi^t(w), \Phi^t(x)) \leq C \exp(\pm \lambda t), \quad t \geq 0. \quad (82)$$

- If moreover \mathcal{V}_0 and \mathcal{F} are of class C^p , $p \geq 1$, and if in addition of the previous conditions the domination inequality, the condition

$$p \|\partial_x X\|_{C^0(\mathbb{R}^m)} \leq \lambda \quad (83)$$

holds, then the functions U , S , U^+ , S^- are of class C^p and

$$\|(U, S)\|_{C^p(\mathbb{R}^m)} \leq C_p \|\mathcal{F}\|_{C^p(\mathbb{R}^{m+2})}. \quad (84)$$

for a suitable constant $C_p > 0$.

- Assume moreover that the vector fields $\mathcal{V}_0, \mathcal{V}$ are R -periodic in x , where R is a lattice in \mathbb{R}^m . Then their flows and the manifolds $\text{Ann}(\mathcal{V})$ and $W^\pm(\text{Ann}(\mathcal{V}))$ pass to the quotient $(\mathbb{R}^m/R) \times \mathbb{R}^2$. Assume that the time-one map of \mathcal{V}_0 on $\mathbb{R}^m/R \times \{0\}$ is C^0 bounded by a constant M . Then, with the previous assumptions, the constant C_p depends only on p , λ and M .

The last statement will be applied in the case where $m = 2\ell$ and $R = c\mathbb{Z}^\ell \times \{0\}$, where c is a positive constant, so that the quotient $\mathbb{R}^{2\ell}/R$ is diffeomorphic to the annulus \mathbb{A}^ℓ .

2. The following result describes the symplectic geometry of our system in the case where \mathcal{V} is a Hamiltonian vector field. We keep the notation of the previous theorem.

The symplectic normally hyperbolic persistence theorem. Endow \mathbb{R}^{2m+2} with a symplectic form Ω such that there exists a constant $C > 0$ such that for all $z \in O$

$$|\Omega(v, w)| \leq C \|v\| \|w\|, \quad \forall v, w \in T_z M. \quad (85)$$

Let \mathcal{H}_0 be a C^2 Hamiltonian on \mathbb{R}^{2m+2} whose Hamiltonian vector field \mathcal{V}_0 satisfies (74) with conditions (75), and consider a Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{P}$. Then there exists a constant $\delta_* > 0$ such that if

$$\|\partial_x X\|_{C^0(\mathbb{R}^{m+2})} \leq \delta_*, \quad \|\mathcal{P}\|_{C^2(\mathbb{R}^{m+2})} \leq \delta_*, \quad (86)$$

the following properties hold.

- The manifold $\text{Ann}(\mathcal{V})$ is Ω -symplectic.
- The manifolds $W^\pm(\text{Ann}(\mathcal{V}))$ are coisotropic and the center-stable and center-unstable foliations $(W^\pm(x))_{x \in \text{Ann}(\mathcal{V})}$ coincide with the characteristic foliations of the manifolds $W^\pm(\text{Ann}(\mathcal{V}))$.
- If \mathcal{H} is C^{p+1} and condition (83) is satisfied, then $W^\pm(\text{Ann}(\mathcal{V}))$ are of class C^p and the foliations $(W^\pm(x))_{x \in \text{Ann}(\mathcal{V})}$ are of class C^{p-1} .
- There exists a neighborhood \mathcal{O} of $\text{Ann}(\mathcal{V})$ and a symplectic straightening symplectic diffeomorphism $\Psi : \mathcal{O} \rightarrow O$ such that

$$\begin{aligned} \Psi(\text{Ann}(\mathcal{V})) &= \mathbb{A}^\ell \times \{(0, 0)\}; \\ \Psi(W^-(\text{Ann}(\mathcal{V}))) &\subset \mathbb{A}^\ell \times (\mathbb{R} \times \{0\}), \quad \Psi(W^-(\text{Ann}(\mathcal{V}))) \subset \mathbb{A}^\ell \times (\{0\} \times \mathbb{R}); \\ \Psi(W^-(x)) &\subset \{\Psi(x)\} \times (\mathbb{R} \times \{0\}), \quad \Psi(W^+(x)) \subset \{\Psi(x)\} \times (\{0\} \times \mathbb{R}). \end{aligned} \quad (87)$$

See [Mar] for a proof.

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